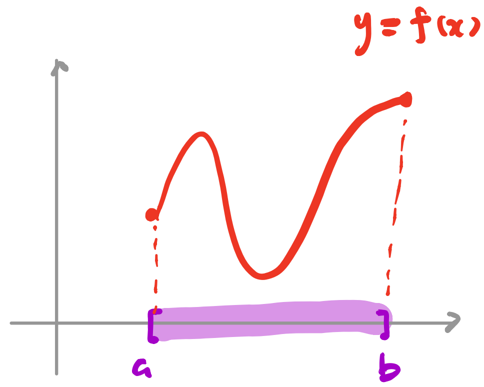
MATH 2050C Lecture on 4/17/2020

[Announcement: PS 10 due next Mon, last two PS's are posted.]

Previously, we study cts $f: A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ is an arbitrary subset.

Q: Can we say more when A is an interval?



Recall: $f: [a,b] \to \mathbb{R}$ continuous.

$\because$ All points $c \in [a,b]$ are cluster pts.

$$\lim_{x \to c} f(x) = f(c) \quad \forall c \in [a,b]$$

$\forall c \in [a,b]$, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, c) > 0$ s.t.

$$|f(x) - f(c)| < \varepsilon \quad \text{for } |x - c| < \delta.$$

A: Two important theorems:

Extreme Value Thm (EVT) ✓ & Intermediate Value Thm (IVT)

Recall: f cts at $c \Rightarrow f$ is "locally bdd" near c.

i.e. $\exists M > 0$, $\exists \delta > 0$ s.t. $|f(x)| \leq M$ for $|x - c| < \delta$.

(M and δ: may depend on c)

When f is defined on a closed & bdd interval, then f is "globally" bounded.

Boundedness Thm: Any cts $f: [a,b] \to \mathbb{R}$ is bdd.

i.e. $\exists M > 0$ s.t. $|f(x)| \leq M \quad \forall x \in [a,b]$.

(M: indep. of c!)

Proof: We argue by contradiction. Suppose NOT.

$\Rightarrow \forall n \in \mathbb{N}$, $\exists x_n \in [a,b]$ s.t. $|f(x_n)| > n$

Note that (x_n) is bdd seq. By Bolzano-Weierstrass Thm,

$\exists$ convergent subseq. (x_{n_k}) of (x_n), say $x_* := \lim(x_{n_k})$.

By Limit Thm, $a \le x_{n_k} \le b \quad \forall k \in \mathbb{N} \implies a \le x_* \le b$

i.e. $x_* \in [a,b]$, so $f(x_*)$ is defined.

Since f is cts on $[a,b]$, in particular, cts at x_*.

Seq. Criteria $\Rightarrow \quad \lim(f(x_{n_k})) = f(x_*)$

So, $(f(x_{n_k}))$ is a convergent seq. hence bdd.

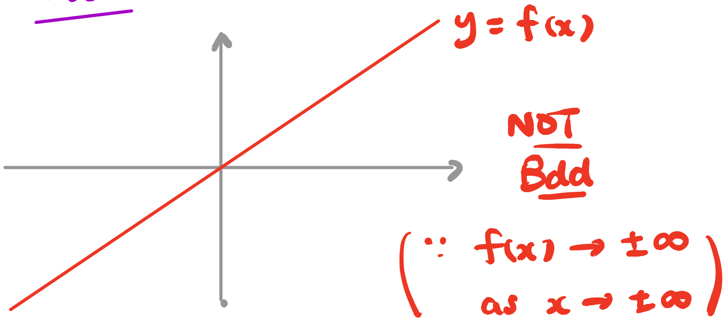
However, by construction $|f(x_{n_k})| > n_k \ge k \quad \forall k \in \mathbb{N}$.

$\Rightarrow (f(x_{n_k}))$ is NOT bdd. — Contradiction. ∎

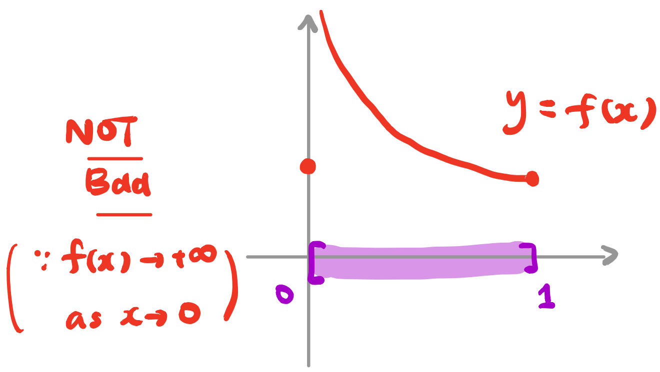
Remark: All the assumptions are necessary!

(1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$ — ($\mathbb{R}$) unbdd!

(2) $f: (0,1] \to \mathbb{R}$, $f(x) := \frac{1}{x}$ — ($(0,1]$) NOT closed!



(3) $f: [0,1] \to \mathbb{R}$, $f(x) := \begin{cases} 1/x & \text{if } x \in (0,1] \\ 1 & \text{if } x = 0 \end{cases}$ — NOT cts at $x=0$!



By Boundedness Thm, we can define:

$$M := \sup\{f(x) \mid x \in [a,b]\}$$

$$m := \inf\{f(x) \mid x \in [a,b]\}$$

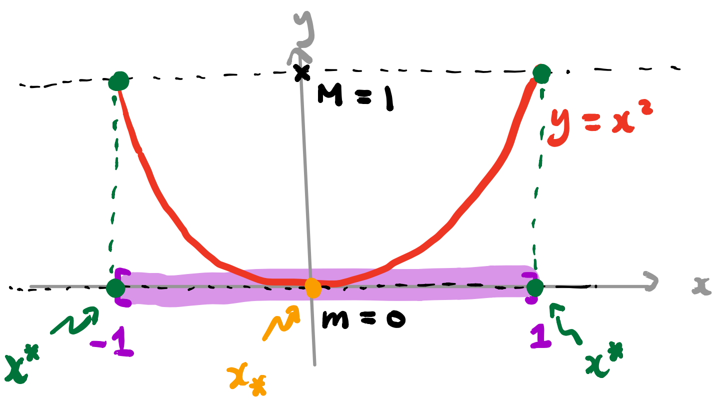
Extreme Value Thm: (EVT)

A cts $f: [a,b] \to \mathbb{R}$ always "achieve" its absolute maximum and absolute minimum.

i.e. $\exists\ x^* \in [a,b]$ s.t. $f(x^*) = M := \sup\{f(x) \mid x \in [a,b]\}$

and $\exists\ x_* \in [a,b]$ s.t. $f(x_*) = m := \inf\{f(x) \mid x \in [a,b]\}$

Example: $f: [-1,1] \to \mathbb{R}$; $f(x) = x^2$

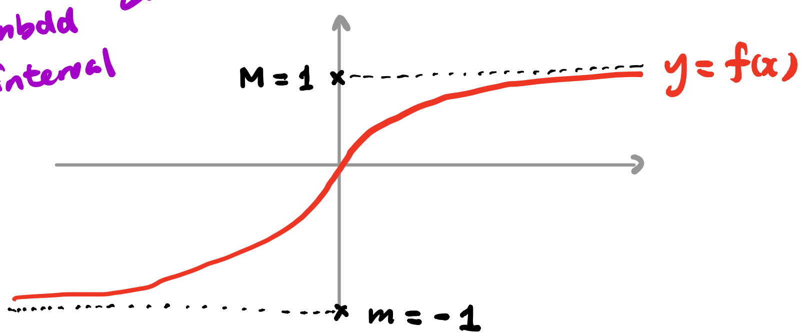


Note: There can be more than one maxima (x^*) and minima (x_*) to make the theorem true.

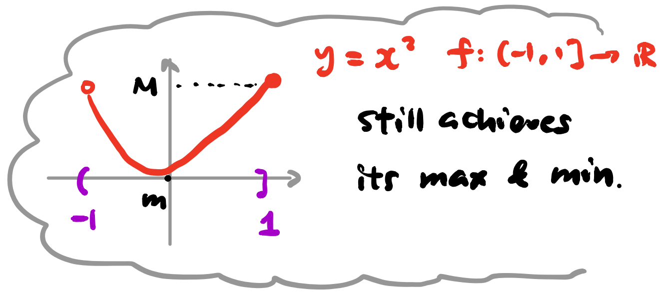
Remarks: ALL assumptions are necessary. (even when f is bdd.)

(1) $f: \mathbb{R} \to \mathbb{R}$; $f(x) = \tanh x$.

unbdd interval

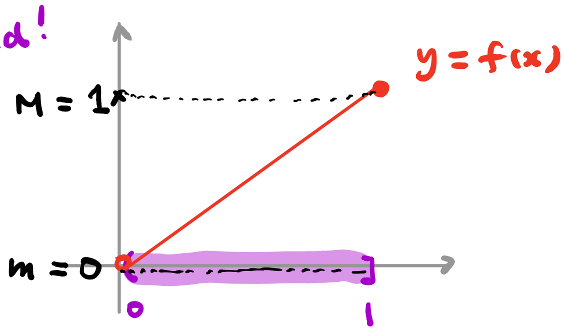


f bdd BuT does NOT achieve its min/max.



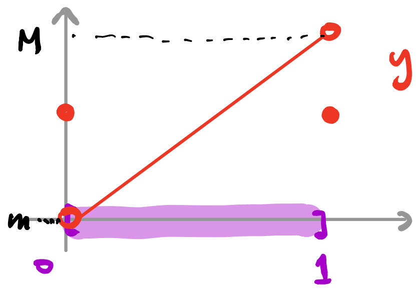
(2) $f: (0,1] \to \mathbb{R}$; $f(x) = x$

NOT closed!



- f bdd
- f achieve its maximum at $x = 1$
- BuT not achieve its minimum.

(3) continuity is needed



Proof of EVT: We just prove

$\exists\, x^* \in [a,b]$ s.t. $f(x^*) = M := \sup\{f(x) \mid x \in [a,b]\}$

Idea: Take limit of "almost" maxima.

Since $M = \sup\{f(x)\}$, $\forall\, \varepsilon > 0$, $\exists\, x_\varepsilon \in [a,b]$ s.t. $M - \varepsilon < f(x_\varepsilon) \le M$

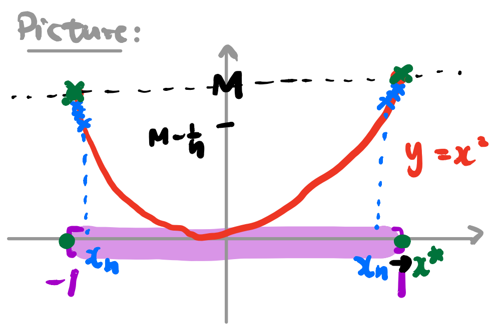
In particular, take $\varepsilon := \frac{1}{n}$ for $n \in \mathbb{N}$, we obtain a seq. (x_n) in $[a,b]$

s.t. $M - \frac{1}{n} < f(x_n) \le M \quad \forall\, n \in \mathbb{N}$

"almost maxima"

As (x_n) is bdd seq, by Bolzano-Weierstrass,

$\exists$ convergent subseq. (x_{n_k}) of (x_n), say $\lim (x_{n_k}) =: x^* \in [a,b]$.



Claim: $f(x^*) = M$.

Pf: Since $M - \frac{1}{n_k} < f(x_{n_k}) \le M \quad \forall\, k \in \mathbb{N}$

take $k \to \infty$, we get $n_k \to \infty$ and

$$M \le \lim f(x_{n_k}) \overset{f \text{ cts at } x^*}{=} f(x^*) \le M.$$

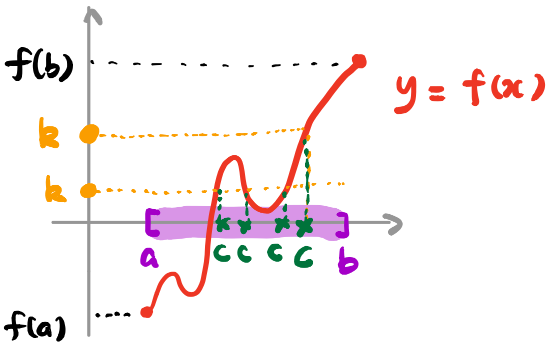
$\square$

Intermediate Value Thm:

Let $f : [a,b] \to \mathbb{R}$ be cts. Suppose $f(a) < f(b)$. (OR $f(a) > f(b)$)

Then, $\forall\, k \in (f(a), f(b))$, $\exists\, c \in (a,b)$ s.t. $f(c) = k$
(OR $(f(b), f(a))$)

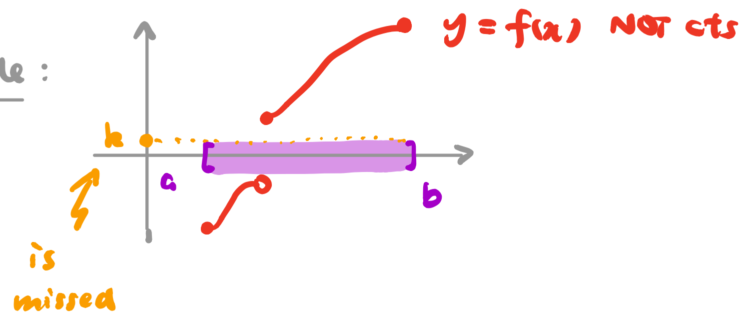
Pictures:



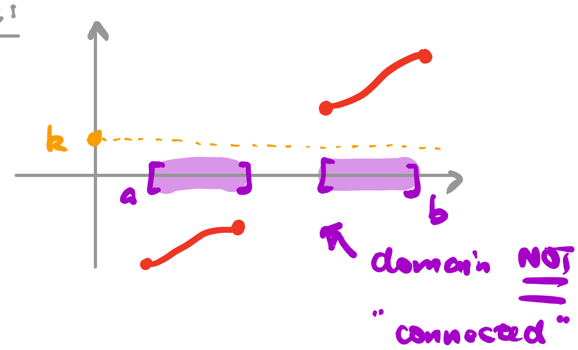
Note: There can be more than one such c's.

BUT All k must be achieved.

Non-example:



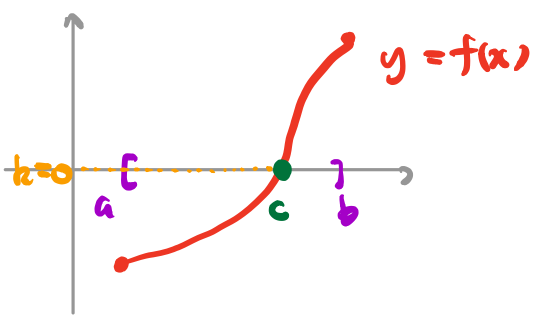
Non-example:



Proof: Idea: Locate "root" i.e. solve $f(x) = 0$.

It suffices (why?) to consider the case:

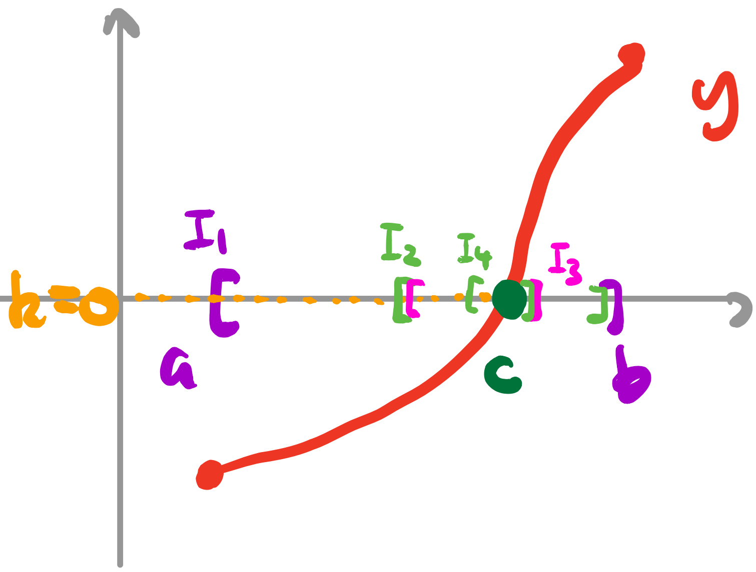
$$f : [a,b] \to \mathbb{R} \text{ cts} \quad \text{s.t.} \quad f(a) < 0 < f(b) \quad \text{and} \quad k = 0$$



Q: How to locate c?

A: Method of Bisection.

Define a nested seq. of closed & bdd intervals I_n as follows:



Take $I_1 := [a,b] = [a_1, b_1]$

· look at the mid-pt $x = \frac{a+b}{2}$

Case 1 : $f(\frac{a+b}{2}) = 0$ DONE!

Case 2 : $f(\frac{a+b}{2}) < 0$

$\Rightarrow$ take $I_2 := [\frac{a_1+b_1}{2}, b_1] = [a_2, b_2]$

Case 3 : $f(\frac{a+b}{2}) > 0$

$\Rightarrow$ take $I_2 := [a_1, \frac{a_1+b_1}{2}] = [a_2, b_2]$

Repeat again to look at the mid-pt of I_2 etc.

Either you get a root at some step. DONE.

Otherwise, you obtain a seq. of closed & bdd intervals $I_n = [a_n, b_n]$

s.t. $I_{n+1} \subseteq I_n \quad \forall n \in \mathbb{N}$

and $f(a_n) < 0 < f(b_n)$.

By Nested Interval Property, we have

$$\bigcap_{n=1}^{\infty} I_n = \{c\} \qquad \because \text{Length}(I_n) \to 0$$

Claim: $f(c) = 0$.

Pf: Since $\lim(a_n) = c = \lim(b_n)$, by continuity at c,

$$f(a_n) < 0 < f(b_n) \quad \forall n \quad \Rightarrow \quad \underbrace{f(c)}_{\lim f(a_n)} \leq 0 \leq \underbrace{f(c)}_{\lim f(b_n)}$$

$\square$

Remark: f cts function on closed & bdd interval.

closed & bdd interval = "compact" & "connected"

EVT, IVT ⟷ cts functions preserve these 2 properties